# Calculus @ QFinance

## Lesson 11

## Tuesday October $9^{\mathrm{th}}$ 2012

## Fubini theorem

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## Product of $\sigma$-algebras

Let $(\Omega_1, \mathcal{A}_1, \mu_1)$ , $(\Omega_2, \mathcal{A}_2, \mu_2)$ two measure spaces.

Put $\Omega := \Omega_1 \times \Omega_2$.

We want to build a measure $\mu$ on $\Omega$ which agrees with the given measures on $\Omega_1$ and $\Omega_2$. Thus we have to provide the domain, i.e. a suitable $\sigma$-algebra on $\Omega$.

The product $\sigma$-algebra $\mathcal{A}$ turns out to be the minimum $\sigma$-algebra on $\Omega$ which contains rectangles $A_1 \times A_2$ with $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$ So this $\sigma$-algebra is generated by the collection of rectangular sets

$$\mathcal{R} = \{A_1 \times A_2 \mid A_1 \in \mathcal{A}_1, \; A_2 \in \mathcal{A}_2\}$$

**Theorem**

Product $\sigma$-algebra $\mathcal{A}_1 \times \mathcal{A}_2$ is the minimum $\sigma$-algebra such that projections

$$\mathrm{Pr}_1 : \Omega \to \Omega_1, \quad \mathrm{Pr}_1(\omega_1, \omega_2) = \omega_1$$

$$\mathrm{Pr}_2 : \Omega \to \Omega_2, \quad \mathrm{Pr}_2(\omega_1, \omega_2) = \omega_2$$

are measurable

The most relevant situation for application is when $\Omega_1 = \Omega_2 = \mathbb{R}$, $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{B}$ Borel $\sigma$-algebra in $\mathbb{R}$.

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**Theorem**

$\sigma$-algebras generated by

$$\mathcal{R} = \{B_1 \times B_2 \mid B_1, B_2 \in \mathcal{B}\}, \quad \mathcal{I} = \{I_1 \times I_2 \mid I_1, I_2 \,\text{intervals}\}$$

are the same.

**Product measure: Guido Fubini Venice 1879, New York 1943**

To build produtct measure we need to work with $\sigma$-finite measure space $(\Omega_1, \mathcal{A}_1, \mu_1)$ , $(\Omega_2, \mathcal{A}_2, \mu_2)$.

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First step: we impose "natural" condition:

$$\mu(A_1 \times A_2) = \mu(A_1)\mu(A_2)$$

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To assign a measure to non rectangular sets we need to introduce the notion of **section** of a subset $A$ of $\Omega_1 \times \Omega_2$

If $A \subset \Omega_1 \times \Omega_2$ and if $\omega_2 \in \Omega_2$ section of foot $\omega_2$ is the subset of $\Omega_1$

$$A_{\omega_2} = \{\omega_1 \in \Omega_1 \mid (\omega_1, \omega_2) \in A\}$$

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**Theorem**

If $A \in \mathcal{A}_1 \times \mathcal{A}_2$ then for any $\omega_2 \in \Omega_2$ we have that $A_{\omega_2} \in \mathcal{A}_1$ and for any $\omega_1 \in \Omega_1$ we have that $A_{\omega_1} \in \mathcal{A}_2$

## Definition

If $A \in \mathcal{A}_1 \times \mathcal{A}_2$ define

$$\mu(A) = \int_{\Omega_2} \mu_1(A_{\omega_2})\, d\mu_2(\omega_2) \tag{$\star$}$$

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If $\mu_1$, $\mu_2$ are $\sigma$-finite measures, then functions

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$$\int_{\Omega_1} \mu_2(A_{\omega_1})\, \mathrm{d}\mu_1(\omega_1) = \int_{\Omega_2} \mu_1(A_{\omega_2})\, \mathrm{d}\mu_2(\omega_2)$$

## Theorem

Set function $\mu$ introduced in $(\star)$ is a measure. It is **unique** since any other measure which coincides with $\mu$ on rectangles is egual to $\mu$ on product $\sigma$-algebra $\mathcal{A}_1 \times \mathcal{A}_2$

## Fubini Theorem on nested integrals

If $f \in \mathcal{L}^1(\Omega_1 \times \Omega_2)$ then functions

$$\omega_1 \mapsto \int_{\Omega_2} f(\omega_1, \omega_2)\, \mathrm{d}\mu_2(\omega_2), \quad \omega_2 \mapsto \int_{\Omega_1} f(\omega_1, \omega_2)\, \mathrm{d}\mu_1(\omega_1)$$

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**Integration in Euclidean spaces. Notations**

$\mathbb{R}^m = \mathbb{R}^p \times \mathbb{R}^q$ with $p + q = m$

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If $A \subset \mathbb{R}^m$ and $x \in \mathbb{R}^p$, the **$x$ section of $A$** is the subset $A_x$ in $\mathbb{R}^q$ defined by:

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If $y \in \mathbb{R}^q$ the ***y* section of *A*** is defined by:

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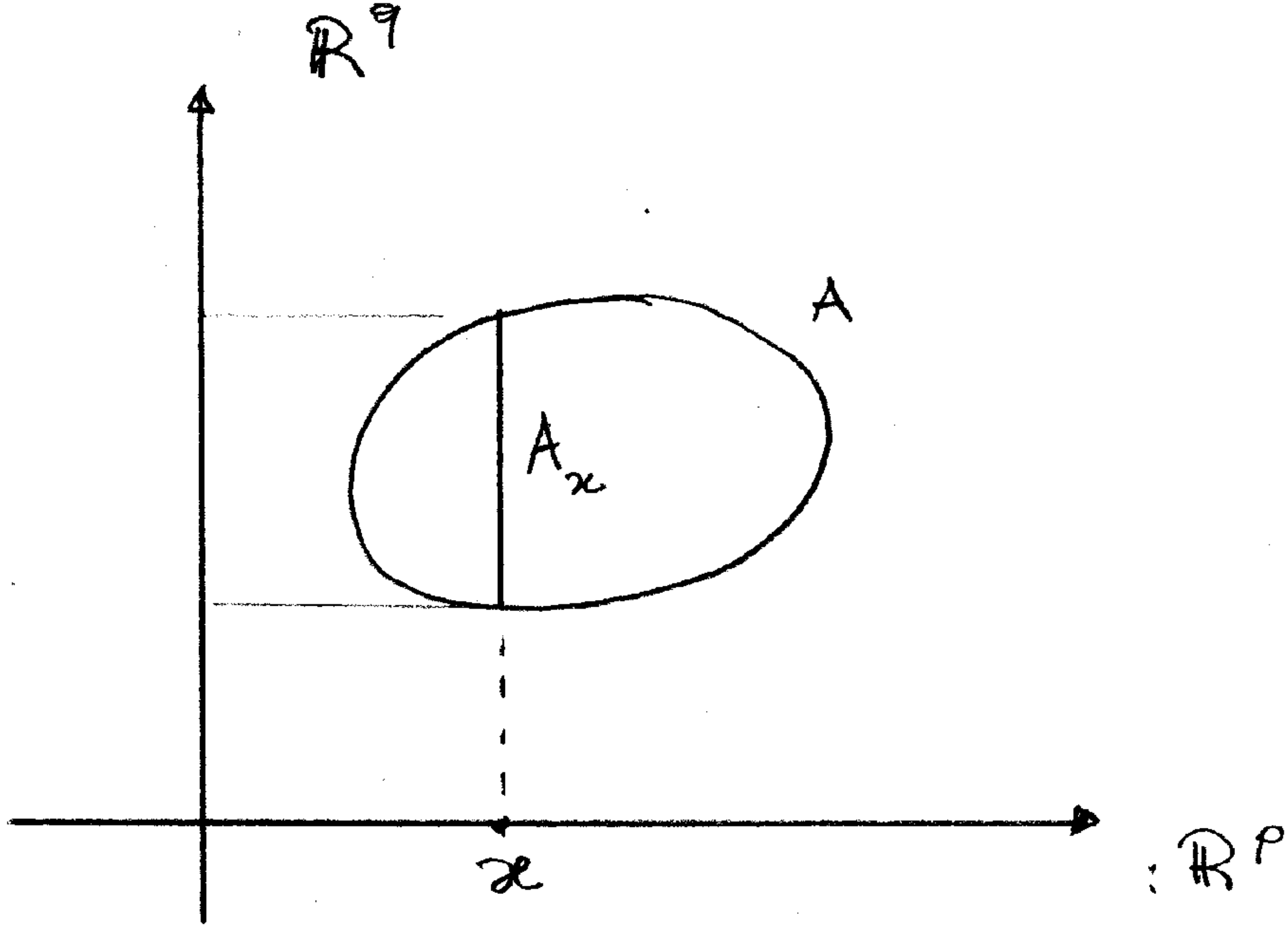
$\mathbb{R}^q$
A
$A_x$
$x$
$\mathbb{R}^p$

## Section's Theorem

Let $A \subset \mathbb{R}^m$ measurable. Then if $x \in \mathbb{R}^p$, section $A_x$ is a.e. measurable. Moreover $x \mapsto \ell_q(A_x)$ is a measurable function and

$$\boldsymbol{\ell_m(A) = \int_{\mathbb{R}^p} \ell_q(A_x) \mathrm{d}\ell_p(x)}$$

$\ell_m$ stands for Lebesgue measure in $\mathbb{R}^m$. The same for $\ell_p$, $\ell_q$

**Fubini's Theorem** G. Fubini: "Il teorema di riduzione per gli integrali doppi" Rend. Sem. Mat. Torino, vo1.9, 1949.

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Assume $A \subset \mathbb{R}^m$ measurable and let $f \in \mathcal{L}(A)$. Define $\mathcal{S}$ as the subset of $\mathbb{R}^p$ where all $q$-section of $A$ have positive measure

$$\boldsymbol{\mathcal{S} = \{x \in \mathbb{R}^p \mid \ell_q(A_x) > 0\}}$$

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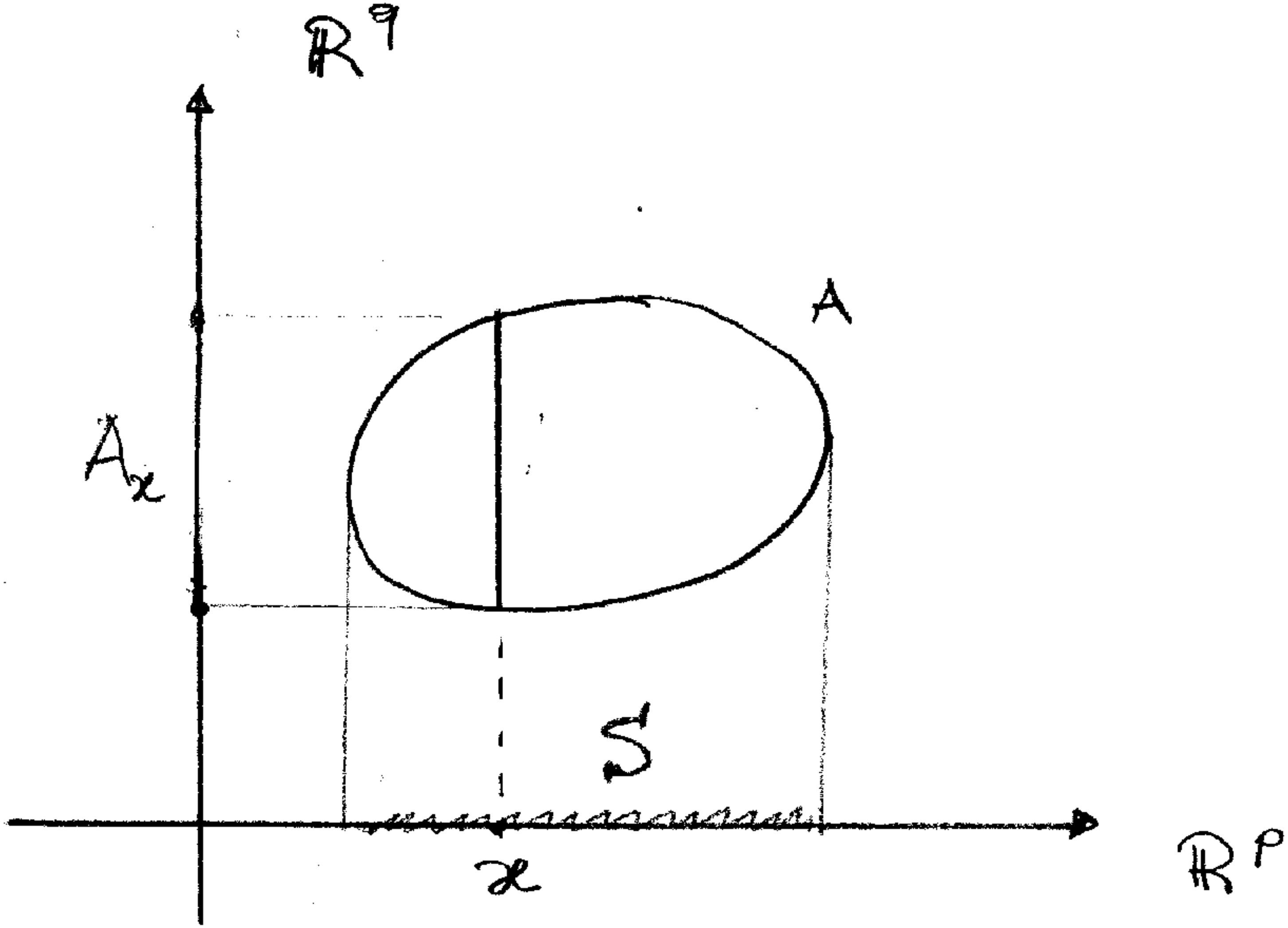
Then the following equality holds

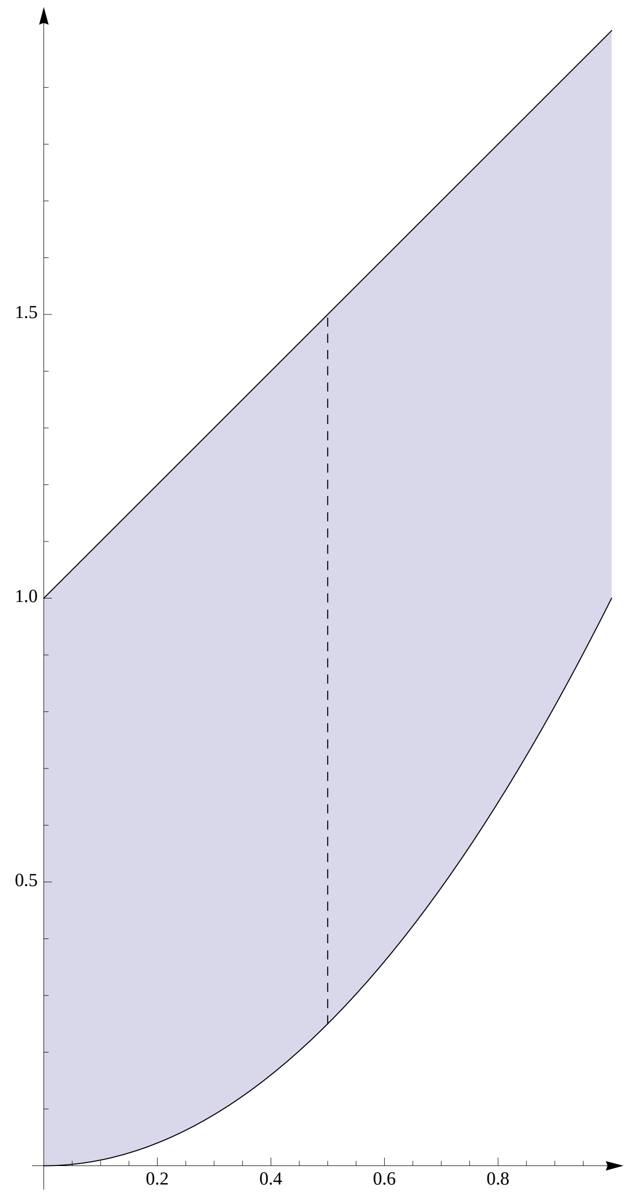
$$\int_{\boldsymbol{A}} \boldsymbol{f(x,y)}\mathbf{d}\boldsymbol{x}\mathbf{d}\boldsymbol{y} = \int_{\boldsymbol{S}} \left( \int_{\boldsymbol{A_x}} \boldsymbol{f(x,y)}\mathbf{d}\boldsymbol{y} \right) \mathbf{d}\boldsymbol{x}$$



$\mathbb{R}^q$
$A$
$A_x$
$S$
$x$
$\mathbb{R}^p$

**Example.** If $A = \left\{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le 1,\ x^2 \le y \le x + 1\right\}$ evaluate:

$$\iint_A xy \, \mathrm{d}x \, \mathrm{d}y$$


1.5
1.0
0.5
0.2
0.4
0.6
0.8

$$\iint_A xy\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \left( \int_{x^2}^{1+x} xy\mathrm{d}y \right) \mathrm{d}x$$

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$$\iint_A xy \, \mathrm{d}x \, \mathrm{d}y = \frac{1}{2} \int_0^1 (-x^5 + x^3 + 2x^2 + x) \, \mathrm{d}x$$

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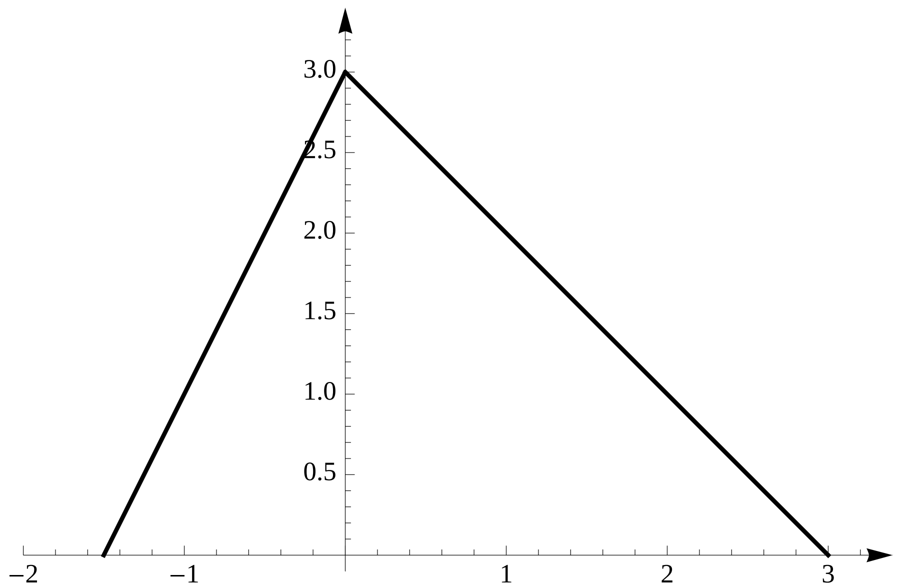
## Example

Given $A = \left\{(x,y) \in \mathbb{R}^2 \mid y \geq 0,\ y \leq -x+3, y \leq 2x+3\right\}$. Evaluate:

$$\iint_A y \,\mathrm{d}x \,\mathrm{d}y$$

Integration domain: triangle with vertices in $(-\frac{3}{2}, 0)$, $(3,0)$, $(0,3)$.



We first integrate in $x$ and then in $y$ :

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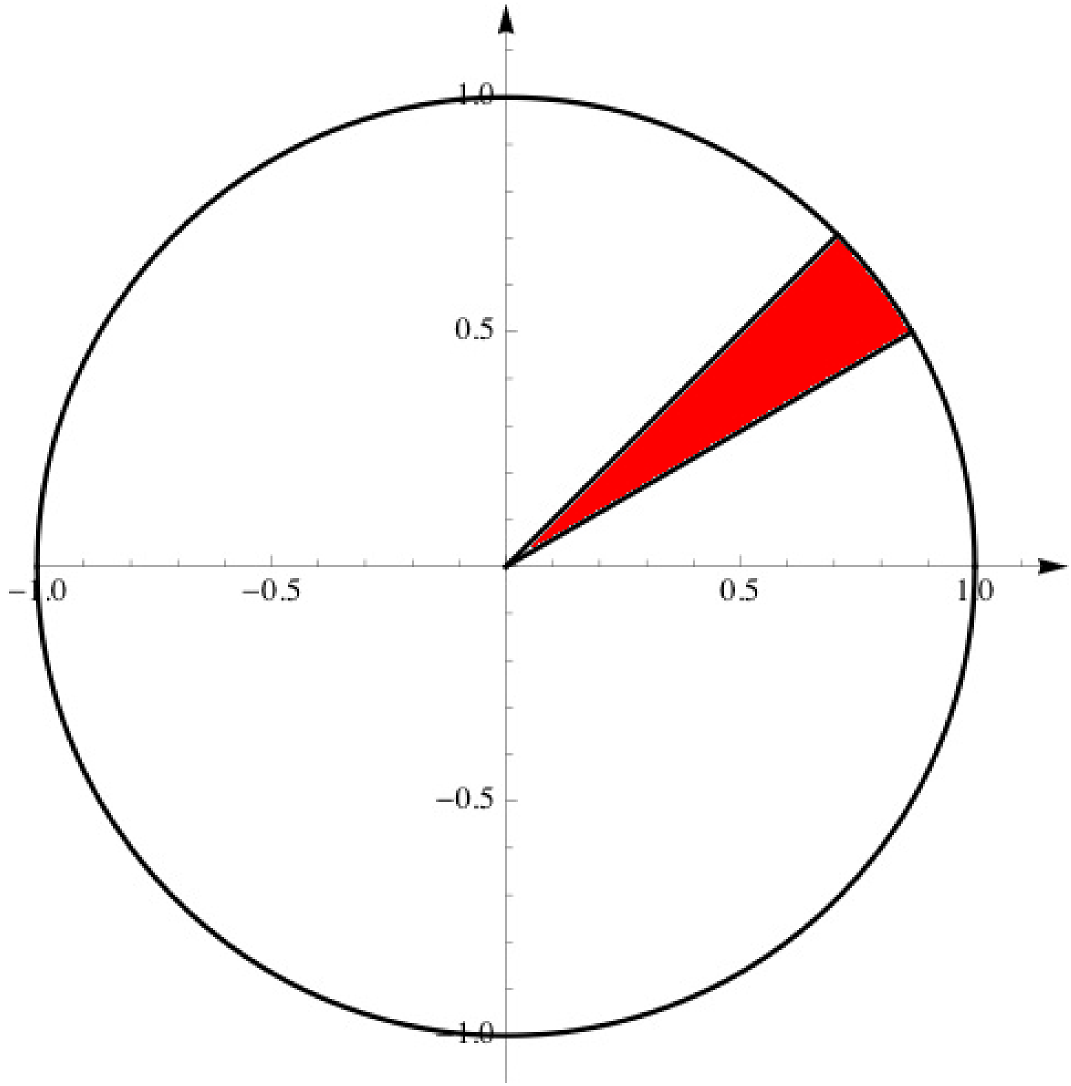
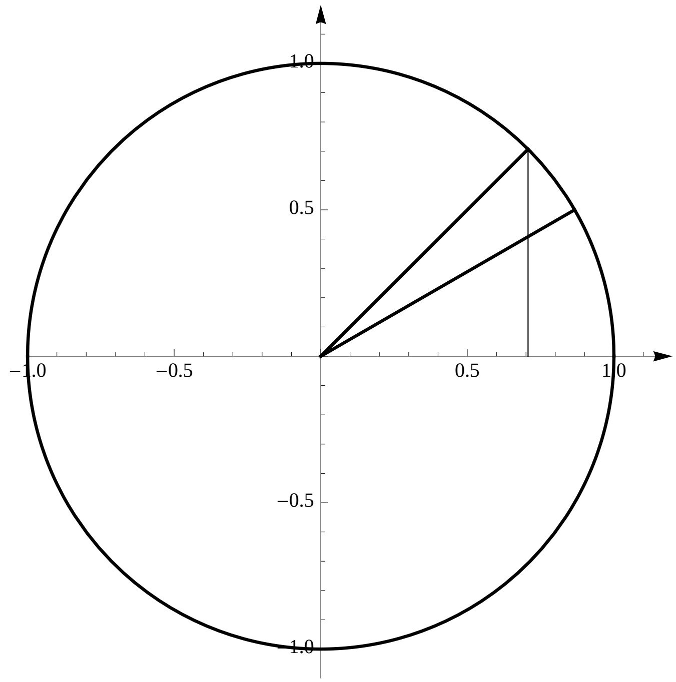
**Example**

Evaluate the measure of

$$A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\} \cap \left\{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq x \leq \sqrt{3}\, y\right\}$$


1.0
0.5
−1.0
−0.5
0.5
1.0
−0.5
−1.0




$$\int_0^{1/\sqrt{2}} \left( \int_{x/\sqrt{3}}^{x} \mathrm{d}y \right) \mathrm{d}x + \int_{1/\sqrt{2}}^{1} \left( \int_{x/\sqrt{3}}^{\sqrt{1-x^2}} \mathrm{d}y \right) \mathrm{d}x$$

For exercise. The only difficult point is

$$\int \sqrt{1-x^2}\,\mathrm{d}x = \frac{1}{2}\left(x\sqrt{1-x^2} + \arcsin x\right)$$

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**Theorem** (E. Cesàro 1881) The probability that two randomly chosen numbers are coprime is given by a product over all primes:

$$\prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^2}\right) = \frac{1}{\sum_{n=1}^{\infty} \frac{1}{n^2}}$$

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G.H. Hardy; E. M. Wright (2008). An Introduction to the Theory of Numbers (6th ed.) Oxford University Press, theorem 332.

**Theorem** (L. Euler)

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

PROOF. (D. Ritelli: to appear on American Mathematical Monthly mid 2013)

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Proof. (D. Ritelli: to appear on American Mathematical Monthly mid 2013)

Let $A = [0, \infty) \times [0, \infty)$ consider

$$\iint_A \frac{\mathrm{d}x\mathrm{d}y}{(1+y)(1+x^2y)}$$

First integrate with respect to $x$ and then with respect to $y$ finding

$$\iint_A \frac{\mathrm{d}x\mathrm{d}y}{(1+y)(1+x^2y)} = \int_0^\infty \left( \frac{1}{1+y} \int_0^\infty \frac{\mathrm{d}x}{1+x^2y} \right) \mathrm{d}y$$

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$$= \int_0^\infty \left(\frac{1}{1+y}\left[\frac{\arctan(\sqrt{y}\,x)}{\sqrt{y}}\right]_{x=0}^{x=\infty}\right)\mathrm{d}y$$

$$= \frac{\pi}{2}\int_0^\infty \frac{\mathrm{d}y}{\sqrt{y}(1+y)} = \frac{\pi}{2}\int_0^\infty \frac{2u}{u(1+u^2)}\mathrm{d}u = \frac{\pi^2}{2}$$

Then we proceed reverting the order of integration:

$$\iint_A \frac{\mathrm{d}x\mathrm{d}y}{(1+y)(1+x^2y)} = \int_0^\infty \left( \int_0^\infty \frac{\mathrm{d}y}{(1+y)(1+x^2y)} \right) \mathrm{d}x$$

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$$\begin{aligned}\iint_A \frac{\mathrm{d}x\mathrm{d}y}{(1+y)(1+x^2y)} &= \int_0^\infty \left(\int_0^\infty \frac{\mathrm{d}y}{(1+y)(1+x^2y)}\right)\mathrm{d}x \\ &= \int_0^\infty \frac{1}{1-x^2}\left(\int_0^\infty \left(\frac{1}{1+y} - \frac{x^2}{1+x^2y}\right)\mathrm{d}y\right)\mathrm{d}x \\ &= \int_0^\infty \frac{1}{1-x^2}\left[\ln\frac{1+y}{1+x^2y}\right]_{y=0}^{y=\infty}\mathrm{d}x \\ &= \int_0^\infty \frac{1}{1-x^2}\ln\frac{1}{x^2}\mathrm{d}x \\ &= \int_0^\infty \frac{\ln x^2}{x^2-1}\mathrm{d}x = \end{aligned}$$

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&= \int_0^\infty \frac{1}{1-x^2}\left(\int_0^\infty \left(\frac{1}{1+y} - \frac{x^2}{1+x^2y}\right)\mathrm{d}y\right)\mathrm{d}x \\
&= \int_0^\infty \frac{1}{1-x^2}\left[\ln\frac{1+y}{1+x^2y}\right]_{y=0}^{y=\infty}\mathrm{d}x \\
&= \int_0^\infty \frac{1}{1-x^2}\ln\frac{1}{x^2}\mathrm{d}x \\
&= \int_0^\infty \frac{\ln x^2}{x^2-1}\mathrm{d}x = \\
&= 2\int_0^\infty \frac{\ln x}{x^2-1}\mathrm{d}x
\end{aligned}$$

Hence, equating we get

$$\int_0^\infty \frac{\ln x}{x^2-1}\mathrm{d}x = \frac{\pi^2}{4}. \tag{a}$$

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Now split the integration domain in (a) between $[0,1]$ and $[1,\infty)$ and change the variable $x = 1/u$ in the second integral, so that

$$\begin{aligned}\int_0^\infty \frac{\ln x}{x^2-1}\mathrm{d}x &= \int_0^1 \frac{\ln x}{x^2-1}\mathrm{d}x + \int_1^\infty \frac{\ln x}{x^2-1}\mathrm{d}x \\ &= \int_0^1 \frac{\ln x}{x^2-1}\mathrm{d}x + \int_0^1 \frac{\ln u}{u^2-1}\mathrm{d}u.\end{aligned} \tag{b}$$

From (a) and (b) we get

$$\int_0^1 \frac{\ln x}{x^2 - 1}\mathrm{d}x = \frac{\pi^2}{8}. \tag{c}$$

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Hence from (c) and (d) follows

$$\sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

To get Euler's statement define $E = \sum_{n=1}^{+\infty} \frac{1}{n^2}$ and split the series in considering even and odd indexes:

$$\sum_{n=1}^{\infty} \frac{1}{(2n)^2} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = E$$

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$$\frac{3}{4} \sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{8} \Longrightarrow \sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Euler 1735: Basel Problem